

# A Principle of Pattern Formation and Recognition

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**Abstract** – The paper proposes a principle of pattern formation and recognition, which meets the requirements of self-contained adaptive control.

## INTRODUCTION

As will be recalled, pattern recognition theory [1, 2, 3] presupposes the use of *a priori* information in the form of a learning set, a classification principle, a class alphabet, and a dictionary of features, among other things. It cannot be doubted, however, that this *a priori* information must, in the general case, be generated as part of a unified process interrelated with recognition. This stems from the general statement of the problem of object control, where the object interacts uniquely and independently with an environment whose properties are little known *a priori*, if at all.

This problem has of late been growing ever more important in practical applications of microprocessor control (in medical equipment, robots, deep-water probes and space probes, etc.), in data-processing systems, in artificial intelligence systems, in the analysis of biological control systems, and in cognition theory.

The general cybernetic issues bearing on the present subject are examined in Ref. [4]. However, the control theory which would fit in with the above statement of the problem now lies outside the author's field of view, except perhaps the control concept set forth in Refs. [5 - 7]. This concept maintains, among other things, that the unique interaction of the controlled object (CO) with the environment makes it necessary to match the pattern formation and recognition (PFR) principle to the action generation and decision-making process.

This paper will attempt to expound the PFR principle proper, matched to the above control concept, while divorcing it from the decision-making process, wherever possible.

### 1. STATEMENT OF THE PFR PROBLEM WITHIN THE FRAMEWORK OF THE NATURAL CONTROL CONCEPT

The statement of the PFR problem in a system satisfying the principles of natural control [7] stems from the formal method of system description [5] and from the principle underlying the organization of control systems [6, 7].

### 1.1. Formalization of System Objects

By *macroobjects* we mean the entities which form the system covering, such as the environment, the controlled object (CO), and the control system (CS), and their unions, intersections, and complements. We will represent macroobjects by directed, weighed, disconnected graphs and subgraphs and define them by named pairs which denote the set of *element*-representing vertices and the set of *effect*-representing connections [5]. Formally, a macroobject is identical to a system element. An element with zero output arity is a *sink*, and an element with zero input arity is a *source*. The law by which a source operates is not defined. The various measures of an action will be called its degrees of freedom (DFs).

We describe each macroobject in the following order: (I) the name and designation of a graph, (II) input connections of the graph, (III) the elements of the object incidental to the latter, (IV) DFs of input effects, (V) DFs of the input effects sensed by the object, (VI) output connections of the object, (VII) the elements of the object incidental to the latter, (VIII) DFs of output effects, and (IX) DFs of the output effects initiated by the object.

1.1.1. (I) *The system or environment in the strict sense*  $\mathbb{U}(V, E)$ ; (II) - (IX) is not defined.

1.1.2. (I) *The environment in the broad sense*  $\mathbb{S}(V', E') \subset \mathbb{U}; V' \subset V; E' \subset E$ . (II)  $\mathbf{Y}_K(Y_1, Y_2, \dots, Y_j, \dots, Y_K); |\mathbf{Y}_K| = K$ . (III)  $\tilde{Y}_K \subseteq V'$ . (IV)  $\mathcal{Y}_K(\{y_1\}, \{y_2\}, \dots, \{y_j\}, \dots, \{y_k\})$ . (V)  $\{y_j^\Gamma\} \subseteq \{y_j\}; y_{j,\gamma}^\Gamma \in \{y_j^\Gamma\}; v_\gamma = \{y_{j,\gamma}^\Gamma\}$ , where  $\gamma \in (1, 2, \dots, G)$  and  $\Gamma = \{v_\gamma\}$ . (VI)  $\mathbf{X}_M(X_1, X_2, \dots, X_j, \dots, X_M); |\mathbf{X}_M| = M$ . (VII)  $\tilde{X}_M \subseteq V'$ . (VIII)  $\mathcal{X}(\{x_1\}, \{x_2\}, \dots, \{x_i\}, \dots, \{x_M\})$ . (IX)  $\{x_i^\Psi\} \subseteq \{x_i\}; x_{i,\epsilon}^\Psi \in \{x_i^\Psi\}; \psi_\epsilon = \{x_{i,\epsilon}^\Psi\}$ , where  $\epsilon \in (1, 2, \dots, E)$  and  $\Psi = \{\psi_\epsilon\}$ .

1.1.3. (I) *The controlled object*  $\mathfrak{R}(D, F) \subset \mathbb{U}; D \subset V; F \subset E; D \cap V' = \Lambda$  (empty set),  $F \cap E' = \Lambda; D \cup V' = V; F \cup E' \subset E$ . (II)  $\mathbf{X}_M$ . (III)  $\tilde{X}_M'' \subseteq D$ . (IV)  $\mathcal{X}$ . (V)  $\{x_i^X\}; \{x_i^\Psi\} \cup \{x_i^X\} \subseteq \{x_i\}; \{x_{i,\iota}^X\} \in \{x_i^X\}; \chi_\iota = \{x_{i,\iota}^X\}$ , where  $\iota \in (1, 2, \dots, M)$  and  $X = \{\chi_\iota\}$ .

(VI)  $Y_K$ . (VII)  $\tilde{Y}_K \subseteq D$ . (VIII)  $\mathcal{O}_K$ . (IX)  $\{h_j^r\}$ ;  $\{h_j^r\} \cup \{h_j^r\} \subseteq \{h_j\}$ ;  $h_{j,\gamma}^r \in \{h_j^r\}$ ;  $v_j = \{h_{j,j}^r\}$ , where  $j \in (1, 2, \dots, K)$  and  $\Upsilon = \{v_j\}$ .

1.1.4. (I) *The environment in the narrow sense*  $\mathfrak{B}(V'', E'') \subset \mathfrak{U}$ ;  $V'' \subset V$ ;  $E'' \subset E$ . (II)  $Y_L(y_1, y_2, \dots, y_\lambda, \dots, y_L)$ ;  $|Y_L| = L$ . (III)  $\tilde{Y}_L' \subseteq V''$ . (IV)  $\mathcal{F}(\{\varphi_1\}, \{\varphi_2\}, \dots, \{\varphi_\lambda\}, \dots, \{\varphi_L\})$ . (V)  $\{\varphi_\lambda^z\} \subseteq \{\varphi_\lambda\}$ ;  $\varphi_{\lambda,\zeta}^z \in \{\varphi_\lambda^z\}$ ;  $\delta_\zeta = \{\varphi_{\lambda,\zeta}^z\}$ , where  $\zeta \in (1, 2, \dots, z)$  and  $Z = \{\delta_\zeta\}$ . (VI)  $X_P(x_1, x_2, \dots, x_z, \dots, x_p)$ ;  $|X_P| = P$ . (VII)  $\tilde{X}_P'' \subseteq V''$ . (VIII)  $\mathcal{K}(\{\kappa_1\}, \{\kappa_2\}, \dots, \{\kappa_z\}, \dots, \{\kappa_p\})$ . (IX)  $\{\kappa_z^\Sigma\} \subseteq \{\kappa_z\}$ ;  $\kappa_{z,\zeta}^\Sigma \in \{\kappa_z^\Sigma\}$ ;  $\mu_\zeta = \{\kappa_{z,\zeta}^\Sigma\}$ , where  $\zeta \in (1, 2, \dots, R)$  and  $\Sigma = \{\mu_\zeta\}$ .

1.1.5. (I) *The control system*  $\mathfrak{B}(W, C) \subset \mathfrak{R}$ ;  $W \subset D$ ;  $C \subset F$ ,  $W$  and  $C$  are of informative nature. (II)  $X_p$ , where  $x_z \in X_p$  is an *input pole*. (III)  $\tilde{X}_p' \subseteq W$ . (IV)  $\mathcal{K}$ . (V)  $\{\kappa_z^\Xi\}$ ;  $\{\kappa_z^\Xi\} \cup \{\kappa_z^\Xi\} \subseteq \{\kappa_z\}$ ;  $\kappa_{z,\xi}^\Xi \in \{\kappa_z^\Xi\}$ ;  $v_\xi = \{\kappa_{z,\xi}^\Xi\}$ , where  $\xi \in (1, 2, \dots, N)$  and  $\Xi = \{v_\xi\}$ . (VI)  $Y_L$ , where  $y_\lambda \in Y_L$  is an *output pole*. (VII)  $\tilde{X}_L' \subseteq W$ . (VIII)  $\mathcal{F}$ . (IX)  $\{\varphi_\lambda^\Omega\}$ ;  $\{\varphi_\lambda^\Omega\} \cup \{\varphi_\lambda^z\} \subseteq \{\varphi_\lambda\}$ ;  $\varphi_{\lambda,\eta}^\Omega \in \{\varphi_\lambda^\Omega\}$ ;  $\vartheta_\eta = \{\varphi_{\lambda,\eta}^\Omega\}$ , where  $\eta \in (1, 2, \dots, Q)$  and  $\Omega = \{\vartheta_\eta\}$ .

1.1.6. (I) *The sensor unit (SU)*  $\mathfrak{D} \subset \mathfrak{R}$ . (II) through (V) correspond to (II) through (V) in  $\mathfrak{R}$ . (VI) through (IX) correspond to (VI) through (IX) in the environment  $\mathfrak{B}$ .

1.1.7. (I) *An actuator (Act.)*  $\mathfrak{E} \subset \mathfrak{R}$ . (II) through (V) correspond to (II) through (V) in the environment  $\mathfrak{B}$ . (VI) through (IX) correspond to (VI) through (IX) in  $\mathfrak{R}$ .

The laws by which the objects  $\mathfrak{S}$ ,  $\mathfrak{R}$ ,  $\mathfrak{B}$ ,  $\mathfrak{B}$ ,  $\mathfrak{D}$ , and  $\mathfrak{E}$  operate are defined, respectively, by the relations:  $\sigma: \Gamma \rightarrow \Psi$  and  $s: \Upsilon \rightarrow X$ ;  $o: X \rightarrow Y$ ;  $w: Z \rightarrow \Sigma$ ;  $b: \Xi \rightarrow \Omega$  and  $c: \Sigma \rightarrow Z$ ;  $d: X \rightarrow \Sigma$ ;  $e: Z \rightarrow Y$ .

We define the following relations:  $a_1: \Upsilon \rightarrow \Gamma$ ;  $a_2: \Psi \rightarrow X$ ;  $a_3: \Sigma \rightarrow \Xi$ ;  $a_4: \Omega \rightarrow Z$ , and  $\beta: \{\kappa_z^\Sigma\} \rightarrow \{\kappa_z^\Xi\}$ . We will also use other notation, such as  $\chi_i = v_j s$  or  $v_j s \chi_i$ , where  $v_j \in \Upsilon$  and  $\chi_i \in X$ . Time  $T = \{t\}$  is deemed discrete;  $t_1$  marks the instant when the CS begins operating,  $t_p$  is the present (or current) instant, and  $t_f$  marks the final instant or the instant when the CS ceases operating. Further,  $t_1, t_p, t_f \in T$ .

### 1.2. The Controlled Interaction

We call the sequence of routes in  $\mathfrak{U}$  which begins at a source and ends at a sink a *macroprocess*, and call any part of the macroprocess a *process*. A process may include or be a *cycle*.

In Ref. [5], a controlled interaction (CI) is defined as a cycle running through the CS, the Act., the environment  $\mathfrak{B}$ , and the SU. To a CI cycle there correspond morphisms (Fig. 1).

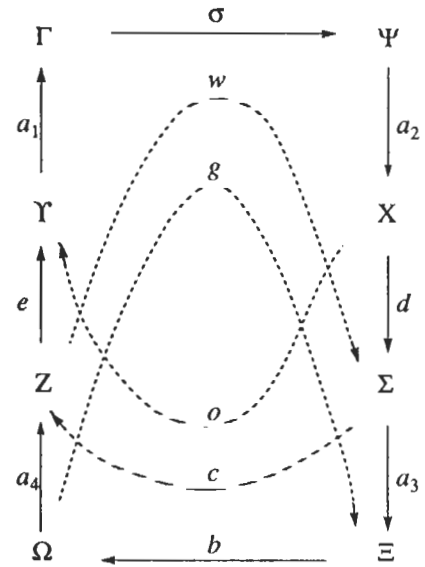


Fig. 1.

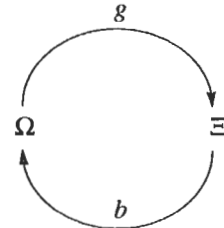


Fig. 2.

If a process contains a number  $n$  of CI cycles, it will be described by the product of relations

$$v_{\xi_1} b a_4 e a_1 \sigma a_2 \chi_{i_1} (os)^{n-1} \chi_{i_2} d a_3 v_{\xi_2}, \quad (1)$$

which must provide a mapping onto itself for each set.

Because all of the "control function" in a system is concentrated in operator  $b$ , we turn to the representation of the system as a pair of macroobjects,  $\mathfrak{B}$  and  $\mathfrak{B}$ , to morphisms (Fig. 2), and to the product of relations  $v_{\xi_1} (bg)^n v_{\xi_2}$  identical to Eq. (1), where  $g: \Omega \rightarrow \Xi$ .

### 1.3. Reflection. Pattern

**Definition 1.** If  $\mathfrak{U}$  and  $\mathfrak{R}$  each has a pair of elements  $s_1$  and  $s_2$ , respectively, such that  $s_1$  is a necessary condition for  $s_2$  to exist, then  $s_2$  is a *reflection* of element  $s_1$  onto  $\mathfrak{R}$ .

**Definition 2.** If  $\mathfrak{U}$  and  $\mathfrak{B}$  each has a pair of elements  $s_1$  and  $s_2$ , respectively, such that  $s_1$  is a necessary condition for  $s_2$  to occur, and satisfying a certain connection index in  $\mathfrak{B}$  is a sufficient condition, then  $s_2$  is a *pattern* of element  $s_1$  in  $\mathfrak{B}$ , and  $s_1$  is the *prepattern* of element  $s_2$  in  $\mathfrak{U}$ . A pattern is the form of existence for a reflection in  $\mathfrak{B}$ .

We call the finite, constant-power set of specialized elements in  $\mathfrak{B}$  intended to store information the memory  $\mathfrak{M}$  of the CS. A memory element is characterized by its state which can vary with time.

The set of memory elements that could be identified with the patterns

$$\mathfrak{B}\mathfrak{D}(O_1, O_2, \dots, O_\omega, \dots, O_k), \quad (2)$$

where  $O_\omega$  is the designation (which we call a *pseudo-identifier* or an *indicator*) of the  $\omega$ -th pattern and  $J$  is a constant is called the memory of patterns we call the set the *memory of formed patterns*  $\mathfrak{B}\mathfrak{D} = \{O_\omega\} \subseteq \mathfrak{M}\mathfrak{D}$  and  $|\mathfrak{B}\mathfrak{D}| = k - \text{var}(t)$  conceived in CS.

### 1.4. The Statement of the PFR Problem

Let there be specified  $\mathbf{X}_p, \mathcal{H}$  and  $\mathfrak{M}$ , and let the law  $g$  be unknown for  $t_1$ . However, the CS establishes a CI in the manner described in Refs. [6, 7] and maintains it for an unspecified length of time. Consider the possibility, under the circumstances, of organizing in the CS an apparatus which could solve the PFR problem automatically. In seeking a solution, we use the approach outlined in Ref. [6], and in reasoning we adhere to the principles laid down in Ref. [7].

## 2. CONTENT AND FORM OF INPUT INFORMATION

For the CS, the source of information about system  $\mathfrak{U}$  (*input information*) is the set  $\mathbf{X}_p$ .

### 2.1. Content of Input Information

The content of input information is determined by the set of macroprocesses in system  $\mathfrak{U}$  incidental to the input poles  $\mathbf{X}_p$ . We partition the macroprocesses into processes, and the latter into routes joining macroobjects (Fig. 3). Now the content of input information is determined by the infinite number of all chain-linked routes terminating at  $\mathbf{X}_p$ .

We denote the set of system elements corresponding to the end of a route by the symbol "h." (for "head"), and the set of system elements corresponding to its start by the symbol "t." (for "tail").

Then for  $\mathbf{X}_p$  we immediately put  $\{h. dx\} = \tilde{\mathbf{X}}_p''$

$$(\{h. yx'\} \cup \{h. \mathfrak{R}x''\}) \subset \{h. dx\} \quad (3)$$

and  $\{h. yx'\} \cap \{h. \mathfrak{R}x''\} \geq \Lambda$ . Also, for  $\mathbf{X}_M$  we indicate that

$$\tilde{\mathbf{X}}_M = \{h. \mathfrak{E}x\} \cup \{h. yx\}, \{h. \mathfrak{E}x\} \cap \{h. yx\} \geq \Lambda,$$

$$\tilde{\mathbf{X}}_M'' = \{t. xy''\} \cup \{t. x\mathfrak{R}''\} \cup \{t. dx\},$$

$$(\{t. xy''\} \cup \{t. x\mathfrak{R}''\}) \cap \{t. dx\} \geq \Lambda,$$

$$\{t. xy''\} \cap \{t. x\mathfrak{R}''\} \geq \Lambda.$$

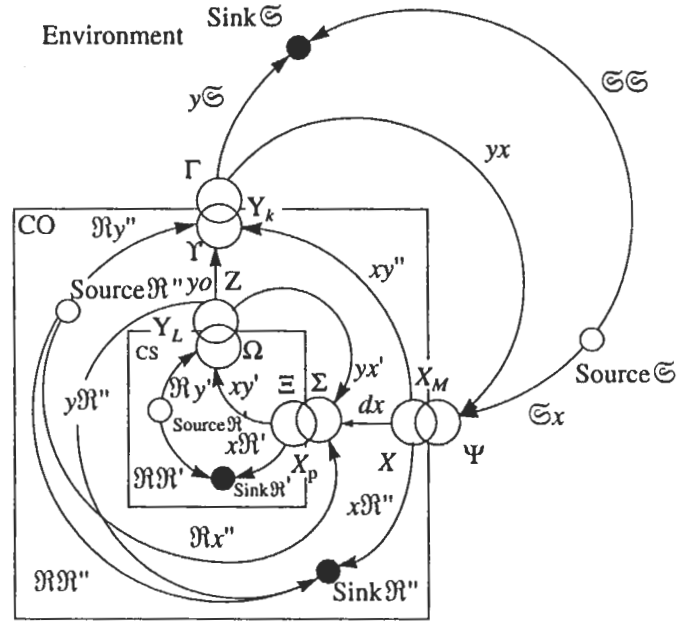


Fig. 3.

It is necessary to consider the relations  $\tilde{\mathbf{X}}_M \xrightarrow{\alpha_1} \mathbf{X}_M$  and  $\mathbf{X}_M \xrightarrow{\alpha_2} \tilde{\mathbf{X}}_M''$ . We assume that  $\mathbf{X}_M \cap \mathbf{X}_p = \Lambda$ . Note that in the CS

$$\tilde{\mathbf{X}}_p' = \{t. xy'\} \cup \{t. x\mathfrak{R}'\}, \quad (4)$$

$$\{t. xy'\} \cap \{t. x\mathfrak{R}'\} \geq \Lambda$$

and take into account the relations  $\tilde{\mathbf{X}}_p'' \xrightarrow{\alpha_3} \mathbf{X}_p$  and  $\mathbf{X}_p \xrightarrow{\alpha_4} \tilde{\mathbf{X}}_p'$ . Note also that  $\tilde{\mathbf{X}}_M'' \cap \tilde{\mathbf{X}}_p'' \geq \Lambda$ ,  $\tilde{\mathbf{X}}_M \cap \tilde{\mathbf{X}}_p' = \Lambda$  and  $(\mathbf{X}_p \cap \mathbf{X}_L) \cup (\mathbf{X}_p \cap \mathbf{X}_M) = \Lambda$ .

As is shown in Ref. [5], there is a nonzero probability for any macroprocess to be influenced by each of the remaining macroprocesses in the system, including  $\mathfrak{E}\mathfrak{E}, \mathfrak{R}\mathfrak{R}'$ , and  $\mathfrak{R}\mathfrak{R}''$ .

The elements  $\tilde{\mathbf{X}}_p'$  are incidental to  $\mathbf{X}_p$  and  $\tilde{\mathbf{X}}_p''$  and, in consequence, to the ends of all processes terminating at  $\tilde{\mathbf{X}}_p'$ . The DFs  $\Xi$  are the attributes of  $\mathbf{X}_p$  and, by the same token, of the elements  $\tilde{\mathbf{X}}_p'$ . In effect, the set  $\Xi$  consists of informational elements belonging to the CS and dependent on input information. Hence and from Definition 1 it follows that the DFs  $\Xi$  are reflections of the environment  $\mathfrak{U}$  on the CS and that any macroprocess can influence input information. Therefore, any element of the environment  $\mathfrak{U}$  can be the prepattern of a pattern.

According to Ref. [5], all system processes sensed by the CS are quasi-deterministic if the system has at least one source.

It is likewise to be realized that the presence of sinks and sources in the system  $\mathfrak{U}$  is solely a consequence of

the constraints imposed on the manner in which the elements and connections of the system  $\mathbb{U}$  are represented.

In summary, the content of input information may be a quasi-deterministic reflection of any processes taking place in the environment  $\mathbb{U}$ .

## 2.2. Localization of the PFR Apparatus

We now define where the PFR apparatus is localized in the system  $\mathbb{U}$ . To begin with, it takes part in control, therefore it belongs to the CI. A pattern is an informational object, therefore the PFR apparatus is located inside the CS and contains the routes  $xy'$ . Because, logically, the PFR task precedes the decision making (action generation) task likewise tackled by the CS, it follows that the PFR apparatus is located closer to the starts of the routes  $xy'$  and the decision making apparatus is located closer to their terminal points (an intersection is quite likely to take place). Obviously, the input poles  $X_p$ , as sources of input information, belong to the PFR apparatus, and the output poles, to the decision making apparatus. Since  $X_p$  are incidental to  $x\mathbb{R}'$ , Eq. (4), it follows that at least some of the routes  $x\mathbb{R}'$  are associated with the PFR apparatus. Some of the routes  $\mathbb{R}y'$  and  $\mathbb{R}\mathbb{R}'$  may likewise belong to the PFR apparatus as, say, causes of interference and information leaks. Nor should we exclude the possibility of any other links existing between  $\mathbb{B}$  and  $\mathbb{B}$  in addition to  $X_p$  and  $Y_L$ , regardless of the level chosen to depict the system.

## 2.3. The Form of Input Information

Input information owes some of its constraints and distortions to its form. In turn, the form of input information is determined by the following CS parameters:  $X_p$ ,  $P$ ,  $\alpha_4$ ,  $\Xi$ ,  $\beta$ , and Eq. (3).

The physical nature of the sensors  $X_p$  and of the DFs  $\Xi$  has a decisive bearing on the filter, the environmental effects it lets reach the CS, the degree to which they are mediated, and their modulation.

Eq. (3) specifies what contributions  $\mathcal{C}$ ,  $\mathbb{R}$ , and  $\mathbb{B}$  make to the content of input information.

It is fundamentally important that the sets  $X_p$  and  $\Xi$  are finite. Because  $|\Xi| < |V \cup E|$ , this inevitably imposes constraints on the content of input information and dictates the principle of operation for the CS [6, 7]. The relation  $\alpha_4$  defines the manner in which input information is quantified.

# 3. THE PATTERN FORMATION AND RECOGNITION PRINCIPLE

## 3.1. Justification of the PFR Principle

We set out to justify the PFR principle expounded in Ref. [6]. As follows from the preceding, every DF  $v_\xi$  can equiprobably be the reflection of an element in  $\mathbb{U}$ , and any element in  $\mathbb{U}$  can be a prepattern. We assign to

each  $v_\xi$  one information bit  $b_\xi \in (0, 1)$  (or a logical event  $b_\xi \in (\text{FALSE}, \text{TRUE})$ ). Further, for  $t \in T$ , we let  $b_\xi = 1$  (or  $b_\xi = \text{TRUE}$ ) if  $v_\xi$  is realized on  $X_p$ , and  $b_\xi = 0$  (or  $b_\xi = \text{FALSE}$ ) if it is not realized.

The set  $\{b_\xi\}$ ,  $\xi = 1, 2, \dots, N$ , contains all possible outcomes from an experiment of interaction between the CS and  $\mathbb{B}$  at time  $t$ . If this experiment yields no

information for the CS, its entropy  $\sum_{\xi=1}^N p_\xi \log_2 (1/p_\xi)$  will take a maximum value,  $\log_2 N$ . But then all outcomes  $b_\xi$  from the experiment will have equal probabilities  $p_\xi$  of occurring. If, on the other hand, the experiment does yield some useful information for the CS, then  $H < \log_2 N$ . This can, however, happen only if the outcomes of the experiment are not equiprobable. Because the content of the input information perceived by the CS is a reflection of  $\mathbb{U}$  in the form of a prepattern, the prepattern is then a non-equiprobable outcome  $b_\xi$  of the experiment at time  $t$  for the CS. Since we have ruled out *a priori* information about  $\mathbb{U}$ , it remains to be supposed that the unequal probabilities of the outcomes  $b_\xi$  expected by the CS at time  $t$  are unequal empirical frequencies  $\mathcal{N}_\xi/k$  of outcomes  $v_\xi$  on a sample of  $k = |(t_1, t_2, \dots, t_p)|$  experiments.

Assuming that the logical events  $b_\xi$  are nonmodal and that the probabilities

$$p_\xi \approx \mathcal{N}_\xi / k \quad (5)$$

are discrete, we find that the CS will perceive a concrete  $v_\xi$  as the reflection of a prepattern in  $\mathbb{U}$  only when  $p_\xi$  exceeds some threshold value

$$\mathcal{N}_\xi / k \geq \mathcal{M}_\omega / k \quad (6)$$

(the meaning of the subscript  $\omega$  will be explained in Eq. (9)).

By choosing a statistically acceptable value for  $p_\xi$ , we can find the respective number  $\mathcal{M}_\omega \approx kp_\xi$ , where  $k$ ,  $1 \leq k \leq |(t_1, \dots, t_p)|$ , is found by trial and error. On eliminating  $k$  from Eq. (6), we conclude that the CS must sense every  $v_\xi$  as a prepattern if the number  $\mathcal{N}_\xi$  of events  $b_\xi = \text{TRUE}$  exceeds the specified fixed value  $\mathcal{M}_\omega$ . The condition

$$\mathcal{N}_\xi = \mathcal{M}_\omega \quad (7)$$

is a necessary condition for forming a pattern for which the prepattern in  $\Xi$  is  $v_\xi$ .

Let us call  $v_\xi$  the *actual* prepattern. It is a reflection of the *true* prepattern distributed in the environment  $\mathbb{U}$ .

The instant  $t_{\omega}^{fo} \in T$  at which Eq. (7) is satisfied is called the *instant of pattern formation* for the prepattern  $v_{\xi}$ . There may be no  $t_{\omega}^{fo}$  for the pattern with the subscript  $\omega$ .

The instant  $t_{\omega}^{fo}$  when  $b_{\xi}^t = \text{TRUE}$  and

$$n_{\xi} > m_{\omega} \tag{8}$$

is the *instant* at which the pattern of the prepattern  $v_{\xi}$  is *recognized*. If  $n_{\xi} > m_{\omega}$  and  $b_{\xi}^t = \text{FALSE}$ , then the pattern of the prepattern  $v_{\xi}$  has been formed but has yet to be recognized.

We now turn to sufficient conditions for the pattern of the prepattern  $v_{\xi}$  to be formed and recognized.

Obviously, the formed pattern  $O_{\omega}$  of a prepattern  $v_{\xi}$  must belong to  $\mathfrak{B}\mathfrak{D}$  and there must exist the mapping

$$fo : \Xi \rightarrow \mathfrak{M} \text{ or } v_{\xi} fo O_{\omega} \tag{9}$$

We denote by  $O'_{\omega} = \text{TRUE}$  the event which consists in that the pattern  $O_{\omega}$  is recognized at time  $t$ ; otherwise, we denote it by  $O'_{\omega} = \text{FALSE}$ . We introduce a condition  $l'_{\omega}$  such that  $l'_{\omega} = \text{TRUE}$  if the image  $O_{\omega} = v_{\xi} fo$  has been recognized, that is, if  $n_{\xi} \geq m_{\omega}$ , and  $l'_{\omega} = \text{FALSE}$ , otherwise.

Then the relation  $fo$  can be expressed in terms of two functions. One is the conjunction

$$O'_{\omega} = b_{\xi}^t \& l'_{\omega}, \quad O'_{\omega} \in (\text{TRUE}, \text{FALSE}). \tag{10}$$

The other,  $f$ , establishes the taxonomic relation between  $\xi$  and  $\omega$

$$\omega = \xi f \text{ or } \omega = f(\xi), \tag{11}$$

then  $J \leq N$ .

The prepattern in  $\Xi$  of the pattern  $O_{\omega}$  is  $v_{\xi} \in \{v_{\xi}\}$ , where

$$\{\xi\} = \omega f^{-1}. \tag{12}$$

In view of Eq. (12), we interpret Eq. (10) as  $O'_{\omega} = (\bigvee_{\xi \in \omega f^{-1}} b_{\xi}^t) \& l'_{\omega}$  or, on setting  $b'_{\omega} = \bigvee_{\xi \in \omega f^{-1}} b_{\xi}^t$ , as  $O'_{\omega} = b'_{\omega} \& l'_{\omega}$ , and instead of  $n_{\xi}$  in Eqs. (7, 8) we use

$$n_{\omega} = \sum_{\xi \in \omega f^{-1}} n_{\xi}. \tag{13}$$

Note also that the operating principle of the CS [6, 7] can be interpreted as a purposeful increase by the control system of the empirical frequencies  $n_{\xi}/k$  of the realizations of DFs  $v_{\xi}$  meeting certain criteria.

### 3.2. Parameter Constraints Imposed by the PFR Principle

Let  $f$  be a one-to-one mapping. Then  $\omega$  is the address of the *memory location*  $s_{\omega}$  of the patterns

$\mathfrak{M}\mathfrak{D} \subset \mathfrak{M}$  conceivable in the CS, whereas  $\mathfrak{B}\mathfrak{D} \subseteq \mathfrak{M}\mathfrak{D}$ , and

$$s_{\omega} \begin{cases} \in \mathfrak{B}\mathfrak{D} \text{ when } l_{\omega} = \text{TRUE}, \\ \in \mathfrak{M}\mathfrak{D} \setminus \mathfrak{B}\mathfrak{D} \text{ when } l_{\omega} = \text{FALSE}. \end{cases}$$

Let there be specified all  $m_{\omega}$ ,  $\omega = 1, 2, \dots, N$ ;  $N = 2^p$ . We take a number  $N$  of  $r_{\omega}$ -bit locations  $s_{\omega}$  to store binary numbers, where

$$r_{\omega} = [\log_2 m_{\omega} + 1] \tag{14}$$

and a 0 (FALSE) or a 1 (TRUE) can be written into each bit. Then the size of the memory  $\mathfrak{M}\mathfrak{D}$  is

$$\sum_{\omega=1}^N r_{\omega} = \sum_{\omega=1}^{2^p} [\log_2 m_{\omega} + 1] \text{ bits.}$$

At each time  $t$ , a binary number  $n'_{\omega}$  is written into each location  $s_{\omega}$  by the following rule:

$$\left\{ \begin{aligned} n'_{\omega} &= 0; \\ n'_{\omega} &= \begin{cases} n'_{\omega} + b'_{\omega} & \text{when } l'_{\omega} = \text{FALSE}, \\ m_{\omega} & \text{when } l'_{\omega} = \text{TRUE}; \end{cases} \\ l'_{\omega} &= \begin{cases} \text{FALSE} & \text{when } n'_{\omega} < m_{\omega}, \\ \text{TRUE} & \text{when } n'_{\omega} \geq m_{\omega}; \end{cases} \\ t_{\omega}^{fo} &= t \text{ when } n'_{\omega} = m_{\omega}. \end{aligned} \right. \tag{15}$$

Then to the set of formed patterns  $\mathfrak{B}\mathfrak{D} = \{O_{\omega}\}$  there corresponds the set  $\{s_{\omega} | l'_{\omega} = \text{TRUE}\}$ .

Because the locations  $s_{\omega}$  are identified by a mapping  $f$  (in the general case, by a mapping  $\{\xi\}$  onto  $\{\omega\}$  and not necessarily a one-to-one mapping), there is no need to expend any of the memory  $\mathfrak{M}$  to hold the addresses of the locations  $s_{\omega}$ . Information retrieval from  $s_{\omega}$  likewise uses a fixed mapping [6, 7]. Recall also that the CS can access  $s_{\omega}$  only in the course of interaction with  $\mathfrak{B}\mathfrak{B}$  via the cycles  $yo \rightarrow yx \rightarrow dx \rightarrow xy'$  or  $yx' \rightarrow xy'$ .

We now consider the case where the empirical frequencies  $c(\xi) = n_{\xi}/k$  are small and comparable with

$1/N$ . That is,  $\mathcal{N}'_{\xi} < \mathcal{M}_{\omega}$  everywhere on  $T$  for the specified  $\mathcal{M}_{\omega}$  and  $T$ . Then by the prepatterns in  $\Xi$  we mean certain subsets  $\{\kappa_{z,\xi}^{\Xi}\}$  of sets  $v_{\xi}$ . We set  $\gamma_{i,\xi} = \{\kappa_{z,\xi}^{\Xi}\} \subset v_{\xi}$ ,  $m_{i,\xi} = |\gamma_{i,\xi}|$ . Let  $\bigcup_{\forall i} \gamma_{i,\xi} \subseteq v_{\xi}$ , and  $\bigcap_{\forall i} \gamma_{i,\xi} \geq \Lambda$ . We further assume that

$$\gamma_{i_1,\xi_1} = \gamma_{i_2,\xi_2} \tag{16}$$

leads to  $i_1 = i_2$  and  $\xi_1 = \xi_2$ . Then  $c(i, \xi) = \mathcal{N}_{i,\xi} / k$  is the empirical frequency of the event  $\beta_{i,\xi} = \text{TRUE}$  on the

sample  $T$ ,  $k = |T|$  and  $\sum_{\xi=1}^N \sum_{i=1}^{N'} c(i, \xi) = 1$ . Let there be

specified the numbers  $\mathcal{M}_{i,\omega}$ . Then by decreasing the value of  $m_{i,\xi}$  from  $m_{i,\xi} = P = |v_{\xi}|$  to  $m_{i,\xi} = 1$ , we find  $m_{i,\xi}$  such that  $\mathcal{N}_{i,\xi} > \mathcal{M}_{i,\omega}$ .

If no such  $m_{i,\xi}$  are found, then, as prepatterns in the system in question, we should take the events  $\beta_{i,\xi} = \text{TRUE}$ , which are even less probable than  $p_{i,\xi} = \mathcal{M}_{i,\xi} / k$ , and decrease the numbers  $\mathcal{M}_{i,\omega}$  as far as  $\mathcal{M}_{i,\omega} = 1$ .

Suppose we have found certain  $m_{i,\xi} > 1$  and  $\mathcal{M}_{i,\omega} > 1$  such that it is legitimate to speak of prepatterns in  $\Xi$ . Then we should have to establish, as in the case of  $f$ , the mapping of  $\{(i, \xi)\}$  onto  $\{\omega\}$ . Note, however, that  $\gamma_{i,\xi} \subset v_{\xi}$  is included in a set  $\{v_{\xi'}\}$  rather than in a single  $v_{\xi}$ , and that  $\xi \in \{\xi'\}$ . The DFs common to  $v_{\xi}$  are  $\kappa_{z,\xi}^{\Xi} = \kappa_{z,\xi}^{\Xi} \in \{\kappa_{z,\xi}^{\Xi}\} | \kappa_{z,\xi}^{\Xi} \in \gamma_{i,\xi}$ , and the remaining  $z$ 's are represented by all possible combinations of  $\kappa_{z,\xi}^{\Xi}$  from the sets  $\{\kappa_{z,\xi}^{\Xi}\} | \kappa_{z,\xi}^{\Xi} \notin \gamma_{i,\xi}$ .

We limit ourselves to the case  $\{\kappa_{z,\xi}^{\Xi}\} = (0, 1)$ ,  $z = \overline{1, P}$ . We interpret  $\kappa_{z,\xi}^{\Xi} = 1$  as the operation of a sensor  $x_z$  on satisfying the condition  $\beta$ , whereas  $\kappa_{z,\xi}^{\Xi} = 0$ , indicates that the sensor has not operated. We take it that a valid signal is a 1 and not a 0. Then

$$\kappa_{z,\xi}^{\Xi} = 1 | \kappa_{z,\xi}^{\Xi} \in \gamma_{i,\xi} \tag{17}$$

Because all  $\kappa_{z,\xi}^{\Xi}$  lying in the domain of the function  $f$  are 1's, then  $f$  is defined on the subscripts  $z$  and  $\xi$ , and not on the values of  $\kappa_{z,\xi}^{\Xi}$ .

Let there be specified  $\gamma_{i,\xi}$ . We will write  $\kappa_z$  instead of  $\kappa_{z,\xi}^{\Xi} = 1$  if, for the specified  $z$ ,  $\kappa_{z,\xi}^{\Xi} \in \gamma_{i,\xi}$ , and  $v_z$  instead of  $\kappa_{z,\xi}^{\Xi} = 1$  or  $\bar{v}_z$  instead of  $\kappa_{z,\xi}^{\Xi} = 0$  if, for the spec-

ified  $z$ ,  $\kappa_{z,\xi}^{\Xi} \notin \gamma_{i,\xi}$ . Then it will be an easy matter to enumerate all  $v_{\xi}$ 's which include  $\gamma_{i,\xi}$ . They are

$$(v_1 \cup \bar{v}_1 \cup \kappa_1, v_2 \cup \bar{v}_2 \cup \kappa_2, \dots, v_z \cup \bar{v}_z \cup \kappa_z, \dots, v_p \cup \bar{v}_p \cup \kappa_p), \tag{18}$$

namely

$$(v_1 \cup \kappa_1, v_2 \cup \kappa_2, \dots, v_p \cup \kappa_p), \\ (v_1 \cup \kappa_1, v_2 \cup \kappa_2, \dots, \bar{v}_p \cup \kappa_p), \\ \dots \\ (\bar{v}_1 \cup \kappa_1, \bar{v}_2 \cup \kappa_2, \dots, \bar{v}_p \cup \kappa_p).$$

It is not convenient to identify  $\gamma_{i,\xi}$  with the pair of subscripts  $(i, \xi)$ . Instead, we will use the inverse function  $\gamma_{\omega} = \omega f^{-1}$  for the purpose, where

$$\gamma_{\omega} = \{\kappa_{z,\xi}^{\Xi}\} \subseteq \bigcap_{\forall \xi'} v_{\xi'}, \tag{19}$$

and  $\{v_{\xi'}\}$  is found from Eq. (18). Then  $f$  is a mapping of  $\{\xi\}$  into  $\{\omega\}$ .

The relations  $|\{v_{\xi'}\}| = 2^{P-m_{\omega}}$ ,  $c(\xi') = \sum_{\forall \xi'} \mathcal{N}_{\xi'} / k \sim |\{v_{\xi'}\}|$  and  $p(\xi') = \sum_{\forall \xi'} p(\xi) \sim |\{v_{\xi'}\}|$ , where  $m_{\omega} = |\gamma_{\omega}|$

imply that, in the case of equal *a priori* probabilities  $p(\xi)$ , Eq. (6), decreasing  $m_{\omega}$  by a factor of  $n$  will increase the probability  $p(\xi')$  of the prepattern  $\gamma_{\omega}$  by a factor of  $2^{m_{\omega}(1-1/n)}$ . Thus, changing from  $f$  as a one-to-one mapping of  $\{\xi\}$  onto  $\{\omega\}$  to  $f$  as a mapping of  $\{\xi\}$  onto  $\{\omega\}$  offers an efficient way to increase the empirical frequencies of prepatterns and, thus, to enhance the probability of pattern formation in the CS.

**Definition 3.** We say that the pattern  $O_a$  more adequately fits the actual prepattern than does the pattern  $O_b$  if the value  $2^{P-m_a}$  corresponding to  $O_a$  is smaller than the value  $2^{P-m_b}$  corresponding to  $O_b$ .

In the general case, the powers  $m_{\omega}$  may be distributed anywhere from  $P$  to 1. For the specified  $X_P$  and  $\Xi$ , one may rate as optimal a partition of  $\Xi$  into  $\gamma_{\omega}$  and a distribution  $m_{\omega}(\omega)$  such as would ensure a maximum number of formed patterns essential for control purposes. In a nonoptimal case, a need will arise for the statistical redundancy of the sets  $\{\gamma_{\omega}\}$ .

### 3.3. Pattern Indication

According to the implication of the signal  $O'_{\omega} = \text{TRUE}$ , its time extent should be defined by the interval  $[t_{\omega}^{\circ}, t_{\omega}^{\ast}]$ , where  $t_{\omega}^{\circ}$  represents the instant when the event  $\beta_{\omega} = \text{TRUE}$  occurs, and  $t_{\omega}^{\ast}$  represents the instant of arrival of the signal  $S'_{\omega} = \text{TRUE}$  which does not

depend on  $b_\omega$  directly and will be defined in Sec. 3.7. In view of the signal  $S'_\omega$ , the value of  $O_\omega$  is defined by the function

$$O_\omega^{'+1} = \neg S'_\omega \& ((b'_\omega \& l'_\omega) \vee O'_\omega). \quad (20)$$

Thus, the recognition of a pattern is discontinued only by the recognition of another pattern (related to the signal  $S_\omega$ ) "ousting" the former, which is quite natural.

### 3.4. Extending the Domain of a Function $f$

#### 3.4.1. Subsets of Inputs

According to Ref. [6], to an event  $O'_\omega = \text{TRUE}$ , the CS makes an action correspond. The action causes information to be transferred over routes  $yx'$ ,  $yo \rightarrow yx \rightarrow dx$  to  $X_p$  or over other routes to sinks. In the former case, the information has above all the meaning of a statement to the effect that the selected action is being executed. Because, in form, it is the same as the action of the environment  $\mathfrak{B}$ , it triggers off another PFR process. Physically, some of the routes  $yx'$  and  $yo \rightarrow yx \rightarrow dx$  may take the form of specialized sensors in the actuator. In contrast to  $yo \rightarrow yx \rightarrow dx$ , the routes  $yx'$  do not extend beyond the limits of the CO and consist of elements belonging to the CO, in particular of one element. In the last case mentioned, this can immediately be the element which serves to indicate the signal  $O'_\omega$ .

Therefore, we assume that  $f$  can be defined on the field  $\{\omega\}$  and that  $\{\xi\} \subset \{\omega\}$ , where the sensors in the SU are described in the same way as patterns but have, say,  $\mathcal{M} = 1$ .

Then, by  $f$ , Eq. (11), we mean the set  $\mathfrak{F}(f_1, f_2, \dots, f_j, \dots, f_j)$  of  $m_j$ -ary functions  $f_j$  each of which is defined on the subset  $\{\omega\}_j \subset \{\omega\}$  of the field and has the value

$$\omega_j \in \{\omega\} \quad (21)$$

in the field, where  $m_j = |\{\omega\}_j|$ . Accordingly,  $|\mathfrak{M}\mathfrak{D}| = |\mathfrak{F}| = |\{\{\omega\}_j\}| = J$ .

We denote the transformation performed by the function  $f_j$  as

$$\{\omega\}_j f_j \omega_j \text{ or } \omega_j = \{\omega\}_j f_j. \quad (22)$$

The set  $\{O_\omega\}$ ,  $\omega \in \{\omega\}_j$ , is the *immediate prepattern* of the pattern  $O_\omega$ , and each pattern  $O_\omega \in \{O_\omega\}$  is the *generator* of the pattern  $O_\omega$ .

Let it be always that

$$\omega_j > \omega \in \{\omega\}_j | (\{\omega\}_j, \omega_j) \in f_j, \quad (23)$$

hence,  $\omega_j \notin \{\omega\}_j$ , and this holds for the routes  $yx'$  consisting of one element. If the output arity of such an element is  $n_j$ , then  $|\{f_k\}| \leq n_j$ , where  $\omega_j \in \{\omega\}_k$ ,  $\{\omega\}_k = \omega_k f_k^{-1}$ ,  $k \in (1, 2, \dots, J)$ ,  $\{f_k\} \subseteq \mathfrak{F}$ .

We further require that  $k_1 \neq k_2$  entail  $\omega_{k_1} \neq \omega_{k_2}$ , although  $\{\omega\}_{k_1} \cap \{\omega\}_{k_2} \geq \Lambda$ .

**Theorem 1.** A pattern fits the actual prepattern in  $\Xi$  more adequately than the generator of its prepattern.

*Proof.* Let there be  $f_e, f_f$ , and  $f_g$  such that  $(\omega_e, \omega_f) f_g \omega_g$ , that is,  $O_{\omega_e}$  and  $O_{\omega_f}$  are the generators of the prepattern for the pattern  $O_{\omega_g}$ . We will repeatedly apply the inverse transformation  $f_j^{-1}$  to  $\omega_j$  until we get the set  $\{\omega\}_j \subset \{\xi\}$ .  
Let

$$\gamma_{\omega_e} = \omega_e f_e^{-1} = (\omega_a, \omega_b) \subset \{\xi\},$$

$$\gamma_{\omega_f} = \omega_f f_f^{-1} = (\omega_c, \omega_d) \subset \{\xi\},$$

$$\begin{aligned} \gamma_{\omega_g} &= \omega_g f_g^{-1} = (\omega_e f_e^{-1} \cup \omega_f f_f^{-1}) \\ &= (\omega_a, \omega_b, \omega_c, \omega_d) \subset \{\xi\}. \end{aligned}$$

Then  $m_{\omega_e} = |\gamma_{\omega_e}|$ ,  $m_{\omega_f} = |\gamma_{\omega_f}|$ ,  $m_{\omega_g} = |\gamma_{\omega_g}| = |\gamma_{\omega_e} \cup \gamma_{\omega_f}|$ , and it follows from  $|\{\nu_\xi\}| = 2^{P-m_\omega}$  that  $N_e = 2^{P-m_{\omega_e}}$ ,  $N_f = 2^{P-m_{\omega_f}}$  and  $N_g = 2^{P-m_{\omega_g}}$ . But, according to Eq. (16),  $\gamma_{\omega_e} \neq \gamma_{\omega_f}$ , therefore,  $|\gamma_{\omega_e}| < |\gamma_{\omega_e} \cup \gamma_{\omega_f}| > |\gamma_{\omega_f}|$  and  $m_{\omega_e} < m_{\omega_g} > m_{\omega_f}$ . In consequence,  $N_e > N_g < N_f$  and, by Definition 3, the pattern  $O_{\omega_g}$  fits the prepattern in  $\Xi$  more adequately than does its generators  $O_{\omega_e}$  and  $O_{\omega_f}$ , as was to be shown.

#### 3.4.2. The Concept of Pattern Order

Let us represent  $\omega_j$ , Eq. (21), by an element  $n_{\omega_j} \in \mathcal{U}$ , of input arity  $m_{\omega_j}$  and of output arity  $n_{\omega_j}$ . Then to  $\mathfrak{F}$  there will correspond an incidence matrix  $n_{\omega_j}$  and a directed subgraph of the graph  $\mathcal{U}$ .

The elements  $(n_{\omega_j} | m_{\omega_j} = 0) \in \tilde{X}'_p$  are the *sources* of the subgraph assigned to the PFR apparatus (we suppose that the input to the CS is different in nature from the connections between the elements  $n$ ) not claimed by other elements  $n$ , and constitute the *output effects from the PFR apparatus*. We will call the elements  $n$  which have no output connections other than the output effects of the PFR apparatus, the *final elements*. We will further call the number of connections in the longest of the routes joining  $n_\omega$  to an element from the set  $\tilde{X}'_p$  the *order of the element  $n_\omega$*  and the *order of the pattern  $O_\omega$* .

The final elements are those of the highest orders. To demonstrate, if the highest order were that of an element  $n_u$  other than a final one, there would be an



element connected to the output of  $n_a$  and the level of  $n_a$  would not be the highest one.

We denote by  $k \in (0, 1, \dots, \mathcal{K})$  the order of the element  $n_{\omega_k}$ , where  $\mathcal{K}$  is the maximum (limiting) order. Then the set  $\{n_{\omega}\}$  will partition into subsets  $\{n_{\omega_k}\}$ . We will call  $\{n_{\omega_x}\}$  the *limiting elements*. A limiting element is a final element, but the converse is not always true. We will call  $\{O_{\omega_x}\}$  the *limiting patterns*.

Because the value of  $O_{\omega}^{l+1}$  depends on the domain of definition at  $t$ , Eq. (20), the level of a limiting pattern determines the number of time moments which the CS would need in order to recognize the limiting pattern after its prepattern in  $\Xi$  has acted, if the limiting pattern had been formed. We will call this time interval the *maximum recognition period*.

**Theorem 2.** Increasing by 1 the input and output arities of each element  $n_{\omega}$  will either increase the order of each element  $n_{\omega}$  or give rise to a new output from the PFR apparatus, or both.

*Proof.* We take any element  $n_j$  and one of the connections  $(n_a, n_b)$  included in the route which defines the order of the element  $n_j$ . We increase by 1 the  $m_{\omega}$  and  $n_{\omega}$  of each  $n_{\omega}$ . According to Eq. (23),  $a < b$ , and  $b - a$  is a finite integer. This adds an output connection  $(n_a, n_x)$  to  $n_a$  and an input connection  $(n_y, n_b)$  to  $n_b$ . If  $n_x \equiv n_y$ , then the order of  $n_j$  is increased by 1, and  $a < x = y < b$ . If, on the other hand,  $n_x \neq n_y$ , then the connection  $(n_a, n_x)$  may or may not be an output from the PFR apparatus. If it is not, a new connection  $(n_x, n_{x_1})$  appears. If  $n_{x_1} \equiv n_y$ , the order of the element  $n_j$  is increased by 2, and  $a < x < x_1 = y < b$ . If, on the other hand,  $n_{x_1} \neq n_y$ , the connection  $(n_x, n_{x_1})$  may or may not be an output from the PFR apparatus. If it is not, a new connection  $(n_{x_1}, n_{x_2})$  appears, and so on. Since  $a, b, x, x_1, \dots, y$  are finite integers, then, after all numbers from the interval  $[b, a]$  have been exhausted, there will of necessity emerge either a new output from the PFR apparatus or an element  $n_{x_i} \equiv n_y$ , and the order of the element  $n_j$  will be increased. A similar dichotomy will result if we examine each connection of the selected route, as was to be shown.

*Corollary.* If an increase in the input and output arities of all elements  $n_{\omega}$  leaves the number of outputs from the PFR apparatus unchanged, there will be an increase in the order of each element  $n_{\omega}$ , as follows from Theorem 2 by virtue of the elimination of the third.

Let there be specified a number  $J$  of elements  $n_{\omega}$ . We also let all  $m$  be equal and all  $n$  be equal. We further let  $P$  from the  $J$  elements be sources, and  $G$  be the number of output effects generated by the PFR apparatus. Then

$$J(n - m) + mP = G \tag{24}$$

where  $J, n, m, P, G \in (0, 1, 2, \dots)$ . From Eq. (24) it follows that for  $n = m$ ,  $J$  is independent of  $m, P$ , or  $G$ .

Suppose that  $\{n_{\omega}\}$  is partitioned into  $\{n_{\omega_k}\}$  and that all  $n_{\omega_k}$  of the  $k$ -th order have the same  $m_k$  and the same

$n_k$  which may vary from order to order. Then, on denoting by  $J_k = |\{n_{\omega_k}\}|$ ,  $G_k$  the number of outputs from  $\{n_{\omega_k}\}$ , by  $U_k$  the number of output connections of elements of orders 0 through  $(k - 1)$ , not claimed by the  $k$ -th order,  $k = \overline{1, \mathcal{K}}$ , we obtain the system of equations

$$\left\{ \begin{aligned} J_1 m_1 + U_1 + G_0 &= J_0 n_0; \\ J_2 m_2 + U_2 + G_1 &= J_1 n_1 + U_1; \\ J_3 m_3 + U_3 + G_2 &= J_2 n_2 + U_2; \\ &\dots \\ J_k m_k + U_k + G_{k-1} &= J_{k-1} n_{k-1} + U_{k-1}; \\ &\dots \\ J_{\mathcal{K}} m_{\mathcal{K}} + U_{\mathcal{K}} + G_{\mathcal{K}-1} &= J_{\mathcal{K}-1} n_{\mathcal{K}-1} + U_{\mathcal{K}-1}; \\ U_{\mathcal{K}} &= 0; \\ G_0 + G_1 + \dots + G_k + \dots + G_{\mathcal{K}} &= G; \\ J_0 + J_1 + \dots + J_k + \dots + J_{\mathcal{K}} &= \mathcal{K}. \end{aligned} \right. \tag{25}$$

We require that all input connections of any element  $n_{\omega}$  run from different elements. We then obtain a complementary system of equations of the form

$$\left\{ \begin{aligned} m_1 &\leq J_0; \\ m_2 &\leq J_0 + J_1; \\ m_3 &\leq J_0 + J_1 + J_2; \\ &\dots \\ m_k &\leq J_0 + J_1 + \dots + J_{k-2} + J_{k-1}; \\ &\dots \\ m_{\mathcal{K}} &\leq J_0 + J_1 + \dots + J_{\mathcal{K}-2} + J_{\mathcal{K}-1}. \end{aligned} \right.$$

If it is known that among  $U_k$  connections only  $\hat{U}_k$  run from different elements, then we can derive more rigorous conditions

$$\left\{ \begin{aligned} m_1 &\leq J_0; \\ m_2 &\leq \hat{U}_1 + J_1; \\ &\dots \\ m_k &\leq \hat{U}_{k-1} + J_{k-1}; \\ &\dots \\ m_{\mathcal{K}} &\leq \hat{U}_{\mathcal{K}-1} + J_{\mathcal{K}-1}, \end{aligned} \right.$$

where  $\hat{U}_{k-1} \leq J_0 + J_1 + \dots + J_{k-2}$ .



In the general case where  $m_{\omega}$  and  $n_{\omega}$  are the arities of the element  $n_{\omega}$ , instead of Eq. (24) we obtain

$$\sum_{\omega=1}^J (n_{\omega} - m_{\omega}) = G.$$

If, for each  $k$ , we know  $\{n_{i,k}\}$ ,  $i = \overline{1, J_k}$ ,  $J_k = |\{n_{i,k}\}|$  and the arities  $m_{i,k}$  and  $n_{i,k}$ , then system (25) gives way to a system of equations of the form

$$\left\{ \begin{array}{l} \sum_{i=1}^{J_1} m_{i,1} + U_1 + G_0 = \sum_{j=1}^{J_0} n_{j,0}; \\ \sum_{i=1}^{J_2} m_{i,2} + U_2 + G_1 = \sum_{j=1}^{J_1} n_{j,1} + U_1; \\ \sum_{i=1}^{J_3} m_{i,3} + U_3 + G_2 = \sum_{j=1}^{J_2} n_{j,2} + U_2; \\ \dots \\ \sum_{i=1}^{J_k} m_{i,k} + U_k + G_{k-1} = \sum_{j=1}^{J_{k-1}} n_{j,k-1} + U_{k-1}; \\ \dots \\ \sum_{i=1}^{J_{\mathcal{L}}} m_{i,\mathcal{L}} + U_{\mathcal{L}} + G_{\mathcal{L}-1} = \sum_{j=1}^{J_{\mathcal{L}-1}} n_{j,\mathcal{L}-1} + U_{\mathcal{L}-1}; \\ U_{\mathcal{L}} = 0; \\ G_0 + G_1 + \dots + G_k + \dots + G_{\mathcal{L}} = G; \\ J_0 + J_1 + \dots + J_k + \dots + J_{\mathcal{L}} = \mathcal{L} \end{array} \right.$$

and the corresponding set of conditions

$$\left\{ \begin{array}{l} \max m_{i,1} \leq J_0, \quad i = \overline{1, J_1}; \\ \max m_{i,2} \leq \hat{U}_1 + J_1, \quad i = \overline{1, J_2}; \\ \dots \\ \max m_{i,k} \leq \hat{U}_{k-1} + J_{k-1}, \quad i = \overline{1, J_k}; \\ \dots \\ \max m_{i,\mathcal{L}} \leq \hat{U}_{\mathcal{L}-1} + J_{\mathcal{L}-1}, \quad i = \overline{1, J_{\mathcal{L}}}. \end{array} \right.$$

Let  $E$  be the number of  $n_{\omega}$  which accept outputs from the PFR apparatus,  $E \leq J$ . Then it is expedient (see Sec. 3.1) to put  $G = rE$ ,  $r = 1, 2, \dots$ , thus unifying all output effects from the PFR apparatus.

Since to each  $n_{\omega_j}$  there corresponds  $f_j \in \mathcal{F}$  (as  $j$  and  $\omega_j$  are connected by a one-to-one mapping, we let  $\omega_j = J$ ), then  $\mathcal{F}$  can be partitioned into subsets  $\mathcal{F}_k \subseteq \mathcal{F}$  of functions of the same order,  $\mathcal{F}_k = \{f_k\}$ ,  $\mathcal{F} = \bigcup_{k=0}^{\mathcal{L}} \mathcal{F}_k$ . The subsets  $\{\omega_k\}$  and  $\{O_{\omega_k}\}$  are defined in a similar manner. We denote  $n_{\omega_k} \in \{n_{\omega_k}\}$ ,  $\omega_k \in \{\omega_k\}$ , and  $O_{\omega_k} \in \{O_{\omega_k}\}$ . Then Eq. (22) may be recast as

$$\{\omega\}_k \mathcal{F}_k \{\omega_k\}, \tag{26}$$

where  $\{\omega\}_k = \{\omega_0\} \cup \{\omega_1\} \cup \dots \cup \{\omega_{k-1}\}$ .

### 3.5. Variations in the Function $f$ during the Operation of the Control System

From Eq. (22) it follows that  $O'_{\omega_j} = \bigwedge_{\omega \in \{\omega\}_j} O'_{\omega}$ .

The formation of the pattern  $O_{\omega_j}$  may be interpreted as the achievement of a certain statistical confidence, in that its prepattern  $\bigwedge_{\omega \in \{\omega\}_j} O_{\omega}$  is a nonrandom object in the system  $\mathbb{U}$ . Past that point, the statistical confidence in the prepattern in  $\mathbb{U}$  is increased each time the pattern  $O_{\omega_j}$  is recognized. The excess confidence in the prepattern thus acquired must be utilized to overcome noise in the prepattern.

We set

$$\chi'_j = |\{O'_{\omega} = \text{TRUE}\}|, \tag{27}$$

where  $\omega \in \{\omega\}_j$ , and require that from  $O'_{\omega_j}$  there should follow

$$\chi'_j / m_j \geq \rho_{j,t}, \tag{28}$$

where

$$\rho_{j,t} = \rho(\mathcal{N}'_{\omega}), \tag{29}$$

$$\mathcal{N}'_{\omega} = |\{O'_{\omega} = \text{TRUE}\}| \tag{30}$$

( $t$  runs through the values  $t_1, t_2, \dots, t_p$ ) is the same as Eq. (13), and  $\rho$  is a function which decreases from  $\rho(0) = 1$  to  $\rho(\infty) = \rho_{\min}$ ,  $0 < \rho_{\min} < 1$ .

**Lemma 1.** Given large  $\mathcal{N}_{\omega}$ , low values of  $\rho(\mathcal{N}_{\omega})$  may be allowed for the same false alarm probability (which fact enables the CS to recognize a cat if only the cat's tail can be seen).

*Proof.* Let  $\hat{k} = |(t_1, t_2, \dots, t_p)|$ . Then, according to Eq. (5),  $p_{\xi} = \mathcal{N}_{\xi} / \hat{k}$ . In view of  $\rho(\mathcal{N}_{\omega})$ , we get  $p_{\omega} \sim \mathcal{N}_{\omega} \rho(\mathcal{N}_{\omega}) / \hat{k}$ . If  $1 - p_{\omega} = 1 - p$  is the allowable false alarm probability in pattern recognition, then  $\mathcal{N}_{\omega} \rho(\mathcal{N}_{\omega}) \sim p \hat{k}$  is the equation of a constant-probability line. This brings us to the statement of the lemma.

As will be shown later (Theorem 3, Lemmas 2 - 6 and 11 - 13), a decrease in  $\rho(\mathcal{N}_{\omega_j})$  strongly affects the way some important processes proceed in the CS.

If  $\rho_{j,t}$  is specified in advance, then in Eq. (20) we set

$$b_{\omega_j} = \begin{cases} \text{TRUE (if } \chi'_j / m_j \geq \rho_{j,t} \text{)} \\ \text{FALSE otherwise.} \end{cases} \quad (31)$$

We refer the condition  $\chi'_j / m_j \geq \rho_{j,t}$  to the function  $f_j$ , thus introducing  $\mathcal{N}_{\omega_j}$  in its domain, and obtain  $\omega_j = f_j(\{\omega\}_j, \mathcal{N}'_{\omega_j})$ .

Resorting to Eq. (31) imparts fundamentally new properties to the CS, but calls for more memory space. To demonstrate, the location  $s_{\omega} \in \mathcal{M}\mathcal{D}$  must now store not the number  $\mathcal{M}_{\omega_j}$ , but the number  $\mathcal{N}_{\omega_j}$  and, perhaps,  $\mathcal{N}_{\omega_j} > \mathcal{M}_{\omega_j}$ . Therefore, instead of Eq. (14) we use  $r_{\omega} = \lceil \log_2 \mathcal{N}_{\max} + 1 \rceil$ , where  $\mathcal{M}_{\omega_j} \leq \mathcal{N}_{\max} \leq |T|$ .

### 3.6. Prepattern Switch-Off

According to Refs. [6, 7], to each pattern  $O_{\omega} \in \mathfrak{B}\mathcal{D}$  the Knowledge Base (KB) of the CS makes correspond a certain action. Because a pattern is characterized by how adequately it fits the actual prepattern (Definition 3), both the concept and the measure,  $2^p - m$ , of the adequacy to the actual prepattern may be extended to the action corresponding to that pattern as well. It is this meaning that will be associated with the concept of the adequacy in the subsequent discussion (another interpretation of this concept is given in Ref. [6]). From Theorem 1 it follows that the action corresponding to a pattern fits the actual prepattern more adequately than do the actions corresponding to its generators.

If, in Eq. (20),  $S'_{\omega_{j,k-1}} = b'_{\omega_{j,k}} \& l'_{\omega_{j,k}}$ , this will switch off the immediate prepattern of the recognized pattern along with the associated less adequately fitting actions. Moreover, we require that the time interval from  $t$  to  $t + 2$  be too small for any actions corresponding to the generators to be initiated. Instead, an action will be executed which fits the actual prepattern more adequately. To generate this action, however, a specified correlation must be achieved between the events of simultaneously recognizing the prepattern of the new pattern and executing an adequately fitting action.

Suppose that  $\mathcal{Z}'_{\omega_{j,k}}$  is the number of signals  $b'_{\omega_{j,k}} \& l'_{\omega_{j,k}} = \text{TRUE}$ . For each  $n_{\omega_{j,k}}$  we introduce a constant  $\mathcal{L}_{\omega_{j,k}}$  and a quantity  $\hat{l}'_{\omega_{j,k}}$  such that

$$\hat{l}'_{\omega_{j,k}} = \begin{cases} \text{FALSE, when } \mathcal{Z}'_{\omega_{j,k}} < \mathcal{L}_{\omega_{j,k}}, \\ \text{TRUE, when } \mathcal{Z}'_{\omega_{j,k}} \geq \mathcal{L}_{\omega_{j,k}}, \end{cases} \quad (32)$$

where  $\hat{l}'_{\omega_{j,k}} = \text{FALSE}$ ,  $\mathcal{Z}'_{\omega_{j,k}} = 0$ ,

$$\mathcal{Z}'_{\omega_{j,k}} = \begin{cases} \mathcal{Z}'_{\omega_{j,k}} + b'_{\omega_{j,k}} \& l'_{\omega_{j,k}} \text{ when } \hat{l}'_{\omega_{j,k}} = \text{FALSE}, \\ \mathcal{L}_{\omega_{j,k}} \text{ when } \hat{l}'_{\omega_{j,k}} = \text{TRUE}, \end{cases}$$

and  $b'_{\omega_{j,k}} \& l'_{\omega_{j,k}} \in (0, 1) \equiv (\text{FALSE}, \text{TRUE})$ . Then

$$S'_{\omega_{j,k-1}} = b'_{\omega_{j,k}} \& l'_{\omega_{j,k}} \& \hat{l}'_{\omega_{j,k}}.$$

When an immediate prepattern is switched off, information about it is not lost, because the inverse relation  $\omega_{j,k} f_{j,k}^{-1} = \{\omega_{j,k-1}\}$  is independent of prepattern switch-off.

### 3.7. The Pattern Formation and Recognition Process

#### 3.7.1. Evolution of the PFR Process with Time

We denote by  $\mathcal{Z}'_{\omega}$  the transformation in Eq. (20)

$$\mathcal{Z}'_{\omega} \equiv \lceil S'_{\omega} \& ((b'_{\omega} \& l'_{\omega}) \vee O'_{\omega}) .$$

Since to each  $n_{\omega}$  there corresponds a  $\mathcal{Z}'_{\omega}$  of its own, then, similarly to  $\{n_{\omega}\}$ , we form the set

$$\{\mathcal{Z}'_{\omega_k}\} = \mathfrak{B}_k \quad (33)$$

of  $\mathcal{Z}'_{\omega}$  transforms of the  $k$ -th order. From Eqs. (20), (26), and (33) it follows that

$$\{O'_{\omega}\}_k \mathfrak{B}'_k \{O'_{\omega_k}\}, \quad (34)$$

where

$$\{O'_{\omega}\}_k = \{O'_{\omega_0}\} \cup \{O'_{\omega_1}\} \cup \dots \cup \{O'_{\omega_{k-1}}\}. \quad (35)$$

We denote the form represented by the right-hand side of Eq. (35) by

$$\mathfrak{B}'_k \equiv \{O'_{\omega_0}\} \cup \{O'_{\omega_1}\} \cup \dots \cup \{O'_{\omega_{k-1}}\},$$

and

$$\{O'_{\omega_{k-1}}\} \mathfrak{B}'_k \{O'_{\omega}\}_k.$$

Then Eq. (34) can be rewritten as

$$\{O'_{\omega_{k-1}}\} \mathfrak{B}'_k \{O'_{\omega}\}_k \mathfrak{B}'_k \{O'_{\omega_k}\}.$$

By applying the form  $\mathfrak{B}$  to the values  $\{O'_{\omega_0+1}\}, \dots, \{O'_{\omega_k+1}\}$  of the transforms  $\mathfrak{B}'_0, \dots, \mathfrak{B}'_k$ , we get

$$\{O'_{\omega_{k-1}}\} \mathfrak{B}'_k \{O'_{\omega_k}\} \mathfrak{B}'_k \mathfrak{B}'_{k+1} \{O'_{\omega_{k+1}}\}_{k+1},$$

where

$$\mathfrak{B}'_{k+1} \equiv \{O'_{\omega_0+1}\} \cup \{O'_{\omega_1+1}\} \cup \dots \cup \{O'_{\omega_k+1}\}.$$

In the general form, for a control system having  $\mathcal{K}$  orders, the PFR process can be described by the expression

$$\begin{aligned} &\{O'_{\omega_0}\} \mathfrak{B}'_0 \mathfrak{B}'_1 \{O'_{\omega_1}\} \mathfrak{B}'_1 \mathfrak{B}'_2 \{O'_{\omega_2}\} \mathfrak{B}'_2 \dots \\ &\mathfrak{B}'_k \{O'_{\omega_k}\} \mathfrak{B}'_k \mathfrak{B}'_{k+1} \{O'_{\omega_{k+1}}\} \mathfrak{B}'_{k+1} \dots \quad (36) \\ &\{O'_{\omega_{\mathcal{K}+1}}\}_{\mathcal{K}+1}, \end{aligned}$$

where

$$\{O_{\omega}\}_0 = \{O_{\omega_0}\} \mathfrak{F}_0^{-1}, \quad (37)$$

and  $O_{\omega_0} \in \{O_{\omega}\}_0$  are the generators of an actual prepattern. That is, for  $k=0$ ,

$$O_{\omega_0} = O_{\omega} \equiv \gamma_{\omega} | k=0, \quad (38)$$

where  $\gamma_{\omega}$  is defined in Eq. (19), and, according to Eq. (17),

$$\kappa_{z,\xi}^{\Xi} = \text{TRUE} | \kappa_{z,\xi}^{\Xi} \in \gamma_{\omega} \equiv O_{\omega}.$$

Suppose that, at a time  $t-1$ ,  $O'_{\omega^{-1}} = \text{FALSE}$ ,  $\omega = \bar{1}, J$ , and there exists a certain set of formed patterns. If at times  $t, t+1, t+2, \dots$ , a prepattern  $v_{\xi}$  is realized, the PFR process, Eq. (36), will begin, and will run on for a time not longer than the maximum recognition period, that is, as far as the instant not exceeding  $t+\mathcal{K}$ . By time  $t+\mathcal{K}$ , a certain set of recognized patterns,  $\{O'_{\omega}{}^{t+\mathcal{K}} = \text{TRUE}\}$ , will have been observed. We call  $O'_{\omega}{}^{t+\mathcal{K}} = \text{TRUE}$  the *maximal pattern of the prepattern*  $v_{\xi}$  if  $O'_{\omega}{}^{t+\mathcal{K}} = \text{FALSE}$  for all  $\omega > \max$ . Its order can be found from the equation

$$O_{\max} \in \{O_{\omega_i}\}. \quad (39)$$

We call  $O'_{\omega_0} = \text{TRUE}$  (that is, a sensor signal) the *minimal recognized pattern of the prepattern*  $v_{\xi}$ .

Because  $\mathcal{M} = 1$  for all sensors, then, except the special case of  $v_{\xi} = (0, 0, \dots, 0)$ , both the minimal and the maximal recognized patterns of a prepattern will exist at any time  $t \in T$ .

If, as a prepattern, we take  $\gamma_{\omega} \subset v_{\xi}$ , then  $v_{\xi} \setminus \gamma_{\omega}$  can likewise contain the prepattern at time  $t$ .

**Theorem 3.** If  $\rho_1 > \rho_2$ , Eq. (27), and  $\mathfrak{B}\mathfrak{D}' = \{O_{\omega} = \text{FALSE}\}$ , then  $k_1 \leq k_2$ , where  $k_i$  are the minimal recognized pattern orders corresponding to  $\rho_i$ .

*Proof.* Let for Eq. (34) the following inequality hold true:

$$|\{O_{\omega}\}_k| > |\{O_{\omega_i}\}| \quad (40)$$

and let  $\rho_1$  be specified. We take  $\{O_{v_i} = \text{TRUE}\}_k \subset \{O_{\omega}\}_k$  and let there be found an  $\{O_{v_i}\}$  such that

$$\{O_{v_i}\}_k \mathfrak{F}_k \{O_{v_i}\}. \quad (41)$$

If  $\rho_1 > \rho_2$ , then, in addition to  $O_{v_i}$ , there can exist other  $O_{v_i}$  satisfying Eq. (41) for  $\rho_2$ . Then  $|\{O_{v_i}\}| \leq |\{O_{v_i} \cup O_{v_i'}\}|$ . By applying the form  $\mathfrak{B}$ , we get

$$\begin{aligned} &\{O_{v_i}\} \mathfrak{B}_{k+1} \{O_{v_i}\}_{k+1} \text{ and} \\ &\{O_{v_i} \cup O_{v_i'}\} \mathfrak{B}_{k+1} \{O_{v_i} \cup O_{v_i'}\}_{k+1}. \end{aligned}$$

According to Eq. (41),

$$\{O_{v_i}\}_{k+1} \mathfrak{F}_{k+1} \{O_{v_{i+1}}\} \quad (42)$$

and

$$\{O_{v_i} \cup O_{v_i'}\}_{k+1} \mathfrak{F}_{k+1} \{(O_{v_i} \cup O_{v_i'})_{k+1}\}. \quad (43)$$

As follows from Eq. (40),

$$|\{O_{v_i}\}_{k+1}| > |\{O_{v_{i+1}}\}| \quad (44)$$

and

$$|\{O_{v_i} \cup O_{v_i'}\}_{k+1}| > |\{(O_{v_i} \cup O_{v_i'})_{k+1}\}|. \quad (45)$$

It follows from Eqs. (44) and (45) that there must exist an  $r$  such that in the case of  $k+r \leq \mathcal{K}$

$$|\{O_{v_{k+r}}\}| = 0 \leq |\{(O_{v_i} \cup O_{v_i'} \cup \dots \cup O_{v_r})_{k+r}\}|, \quad (46)$$

whence  $k_1 < k+r, k_2 \geq k+r$  and  $k_1 < k_2$ . If no  $\{O_{v_i}\}$  is found for Eq. (41), then  $k_1 = k_2$ . In consequence,  $k_1 \leq k_2$ , as was to be shown.

**Lemma 2.** Given certain conditions, decreasing the value of  $\rho(\mathcal{N}_{\omega})$  will result in the recognition of the more adequately fitting patterns.

*Proof.* From Eq. (41) of Theorem 3 it follows that  $k_1$  applies to the generators of order- $k_2$  patterns. Hence and also from Theorem 1 and Lemma 1 it follows that with  $\rho_1 > \rho_2 > p\hat{k} / \mathcal{N}_{\xi}$  the more adequately fitting patterns will be recognized without a higher false alarm probability.

**Lemma 3** (without proof). To a more adequately fitting pattern in the KB there corresponds a more adequately fitting action. Since the operation of the KB lies outside the scope of the present paper, no proof will be given for the lemma, especially as it can be deduced from the material presented in Ref. [6]. Here, we limit ourselves to what has been said in Sec. 3.6.

**Lemma 4.** Decreasing  $\rho(\mathcal{N}_\omega)$  creates conditions required for a greater number of patterns to be formed.

*Proof.* As follows from Eq. (46) of Theorem 3, more patterns are recognized when  $\rho_2(\mathcal{N}_\omega) < \rho_1(\mathcal{N}_\omega)$ . Since, however, pattern recognition is a necessary condition for pattern formation, a decrease in  $\rho$  is a prerequisite for a greater number of patterns to be formed. A sufficient condition is a certain number of such recognitions performed.

**Lemma 5.** Decreasing  $\rho(\mathcal{N}_\omega)$  creates the conditions required for the KB of the control system to be expanded.

The statement of Lemma 5 follows immediately from the statement of Lemma 4, because the KB [6] consists of a set of formed patterns Eq. (2) and a corresponding set of generated actions. Also, as is shown in Ref. [6], the formation of new patterns is a necessary condition for new actions to be generated.

**Lemma 6.** Decreasing  $\rho(\mathcal{N}_\omega)$  makes it easier for the CS to adapt itself to the environment.

According to Refs. [6, 7], the CS adapts itself to the environment  $\mathbb{U}$  progressively better as its KB increases in size. The statement of Lemma 6 follows from this observation and from Lemma 5. Also, if a better adapted CS is that whose KB contains more adequately fitting patterns and actions, then the statement of Lemma 6 follows from Lemmas 2 and 3 as well.

3.7.2. The Instantaneous State of the PFR Process

As follows from Eq. (36), the state of the PFR process at time  $t \in T$  is determined by the set

$$\begin{aligned} & \{O'_\omega\}_{\mathcal{K}+1} \\ &= \{O'_\omega\}_0 \cup \{O'_{\omega_0}\} \cup \{O'_{\omega_1}\} \cup \dots \cup \{O'_{\omega_x}\}. \end{aligned}$$

Here, in accord with Eq. (37),  $\{O'_\omega\}_0$  is the set of generators of an actual prepattern. That is, they are not the patterns but the prepatterns of sensor signals.  $\{O'_{\omega_0}\}$  is the set of zero-order patterns (for example, sensor signals).  $\{O'_{\omega_1}\}$  is the set of first-order patterns, etc. Hence, the state of all patterns at time  $t$  is

$$\begin{aligned} & \{O'_\omega\}_{\mathcal{K}+1} \setminus \{O'_\omega\}_0 \\ &= \{O'_{\omega_0}\} \cup \{O'_{\omega_1}\} \cup \dots \cup \{O'_{\omega_x}\}, \end{aligned} \tag{47}$$

where

$$\left\{ \begin{aligned} \{O'_{\omega_0}\} &= \{O'^{-1}_\omega\}_0 \mathfrak{B}'^{-1}_0; \\ \{O'_{\omega_1}\} &= \{O'^{-2}_\omega\}_0 \mathfrak{B}'^{-2}_0 \{O'^{-1}_\omega\} \mathfrak{B}'^{-1}_1 \\ &\quad \times \{O'^{-1}_\omega\}_1 \mathfrak{B}'^{-1}_1; \\ &\quad \dots \\ \{O'_{\omega_k}\} &= \{O'^{-k-1}_\omega\}_0 \mathfrak{B}'^{-k-1}_0 \{O'^{-k}_\omega\} \mathfrak{B}'^{-k}_1 \\ &\quad \times \{O'^{-k}_\omega\}_1 \mathfrak{B}'^{-k}_1 \dots \{O'^{-1}_\omega\} \mathfrak{B}'^{-1}_k \{O'^{-1}_\omega\}_k \mathfrak{B}'^{-1}_k; \\ &\quad \dots \\ \{O'_{\omega_x}\} &= \{O'^{-\mathcal{K}-1}_\omega\}_0 \mathfrak{B}'^{-\mathcal{K}-1}_0 \{O'^{-\mathcal{K}}_\omega\} \mathfrak{B}'^{-\mathcal{K}}_1 \\ &\quad \times \{O'^{-\mathcal{K}}_\omega\}_1 \mathfrak{B}'^{-\mathcal{K}}_1 \dots \{O'^{-1}_\omega\} \mathfrak{B}'^{-1}_{\mathcal{K}} \{O'^{-1}_\omega\}_{\mathcal{K}} \mathfrak{B}'^{-1}_{\mathcal{K}}. \end{aligned} \right. \tag{48}$$

Noting that  $\{O'_{\omega_0}\} = v'_\xi$ , and retaining only the final results of the relation products, we can rewrite Eq. (48) in a more compact form as

$$\left\{ \begin{aligned} \{O'_{\omega_0}\} &= v'_\xi; \\ \{O'_{\omega_1}\} &= \{O'^{-1}_\omega\}_1 \mathfrak{B}'^{-1}_1; \\ &\quad \dots \\ \{O'_{\omega_k}\} &= \{O'^{-1}_\omega\}_k \mathfrak{B}'^{-1}_k; \\ &\quad \dots \\ \{O'_{\omega_x}\} &= \{O'^{-1}_\omega\}_{\mathcal{K}} \mathfrak{B}'^{-1}_{\mathcal{K}}, \end{aligned} \right.$$

where  $\{O'_\omega\}_k$  is the result of Eq. (36).

The right-hand side of Eq. (47) is the instantaneous state of pattern memory, Eq. (2), that is,  $\mathfrak{B}'\mathfrak{D}' = \{O'_{\omega_0}\} \cup \{O'_{\omega_1}\} \cup \dots \cup \{O'_{\omega_k}\} \cup \dots \cup \{O'_{\omega_x}\}$ . The set  $\mathfrak{L}'(\hat{l}'_1, \hat{l}'_2, \dots, \hat{l}'_\omega, \dots, \hat{l}'_k)$  of quantities  $\hat{l}'_\omega$ , Eq. (15), ordered in terms of  $\omega$ , defines the existence of formed patterns at time  $t$ . The set  $\hat{\mathfrak{L}}'(\hat{l}'_1, \hat{l}'_2, \dots, \hat{l}'_\omega, \dots, \hat{l}'_k)$  of quantities  $\hat{l}'_\omega$ , Eq. (32), ordered in terms of  $\omega$ ,

defines the possibility for the prepatterns of patterns  $O_\omega$  to be switched off at time  $t$ .

In consequence, the instantaneous state of the PFR process is described by a triplet of vectors,  $\mathcal{D}'(\mathcal{B}\mathcal{D}', \mathcal{L}', \mathcal{L}')$ . The component elements of the triplet  $\mathcal{D}'$  are described by the following system of relations

$$\left\{ \begin{array}{l} O_{\omega_k}^{i+1} = \neg S_{\omega_k}^i \& ((b_{\omega_k}^i \& l_{\omega_k}^i) \vee O_{\omega_k}^i); \\ S_{\omega_{k-1}}^{i+1} = b_{\omega_k}^i \& l_{\omega_k}^i \& \tilde{l}_{\omega_k}^i; \\ l_{\omega_k}^i = \begin{cases} \text{FALSE, when } n_{\omega_k}^i < m_{\omega_k}; \\ \text{TRUE, when } n_{\omega_k}^i \geq m_{\omega_k}; \end{cases} \\ \tilde{l}_{\omega_k}^i = \begin{cases} \text{FALSE, when } \mathcal{Z}_{\omega_k}^i < \mathcal{L}_{\omega_k}; \\ \text{TRUE, when } \mathcal{Z}_{\omega_k}^i \geq \mathcal{L}_{\omega_k}. \end{cases} \end{array} \right.$$

Thus, on perceiving the actions of the environment  $\mathcal{B}$  as the sequence

$$v_{\xi_1}^1, v_{\xi_2}^2, \dots, v_{\xi_i}^i, \dots, v_{\xi_p}^p; v_{\xi} \in \Xi, \quad (49)$$

the PFR apparatus of the CS transforms it into the sequence

$$\mathcal{D}^1, \mathcal{D}^2, \dots, \mathcal{D}^i, \dots, \mathcal{D}^p. \quad (50)$$

In turn, the sequence in Eq. (50) is transformed into a sequence of actions selected by the process described in Refs. [6, 7].

It is worth noting some important properties of the PFR apparatus.

**Property 1.** The state of the pattern memory  $\mathcal{B}\mathcal{D}$  at time  $t$  depends on what prepatterns of sensor signals,  $\{O_\omega\}_0$ , were arriving throughout the past history from  $t = 1$  to  $t = t_p$ . Indeed, this conclusion about the interval from  $t_p - \mathcal{H} - 1$  to  $t_p$  follows from Eq. (48). But the state at time  $t_p - 1$  depends on the interval from  $t_p - \mathcal{H} - 2$  to  $t_p - 1$ , and so on, as was to be shown.

With time, the recognized patterns of an actual prepattern in  $\{O_{\omega_0}\}$ , Eq. (38), give way to the patterns of a new prepattern in  $\{O_{\omega_0}\}$ , with all previous patterns being ousted during a time likewise equal to the maximum recognition period.

In order that all transients caused by the occurrence of a next prepattern on  $X_p$  can die out, it is essential for this prepattern to stay on  $X_p$  for at least the maximum recognition period. Hence the requirement that the maximum recognition period  $\mathcal{H}\Delta t$  should not exceed

(in a statistical sense) the characteristic time of changes in the environment  $\mathcal{B}$ .

**Property 2.** If the PFR apparatus has at its disposal functions  $f$  of an order higher than the first, then the PFR process generated by a particular prepattern can be influenced by the prepatterns realized on  $X_p$  at previous times, as follows from Eq. (36).

**Lemma 7.** The subscript  $\omega$  and the order  $k$  of the maximal recognized pattern, Eq. (39), of a particular prepattern,  $\{O_{\omega_0}\}$ , depend on the past history of that prepattern, as appears from Property 2.

**Lemma 8.** Three prepatterns  $O_1, O_2$ , and  $O_3$ , consecutively realized on  $X_p$ , can generate a pattern of an order proportional to the duration of the second among these prepatterns.

To see this, suppose the duration of  $O_2$  is  $k\Delta t \geq 0$ . This implies that a time  $k\Delta t$  later information about  $O_1$  can only stay on the  $k + 1$ st order (as the  $k + 1$ st order pattern which might be recognized as the consequence of  $O_1$ ), whereas on orders from 0 to  $k$  it will have been ousted by patterns corresponding to  $O_2$ . By the instant the prepattern  $O_3$  occurs, this information remains only on the  $k + 2$ nd order. If, in  $\mathcal{F}$ , there is found an  $f_\omega$  such that its immediate prepattern includes both  $O_3$  and the  $(k + 2)$ nd-order pattern formed by  $O_1$ , then  $f_\omega$  will form pattern of at least the  $k + 3$ rd order, as was to be shown.

This property can be interpreted as the ability of the CS to form and recognize patterns of regular sequences of events in the system.

**Lemma 9.** As follows from Lemmas 7 and 8, if two or more prepatterns with the same period-to-duration ratio more than  $n_j$  times, then, if  $\mathcal{F}$  contains an appropriate  $f_j$ , a pattern will be generated for which the immediate prepattern is the above sequence of prepatterns. Moreover, the generators of this prepattern will indirectly include a time interval.

Note that a recognized pattern is ousted other than by a change of its prepattern, but in only two cases: either it is switched off as the prepattern of the recognized pattern of a higher order, or it is switched off by a signal similar to  $S_\omega$  as the prepattern of an action executed by the CS. The latter mechanism is the only one for the final elements  $n_\omega$ , and it is an alternative one for those  $n_\omega$  whose outputs include the outputs of the PFR apparatus. This adds an argument in favor of considering the PFR process along with the decision-making process.

#### 4. MODELING IN THE PFR APPARATUS

##### 4.1. Sections of the PFR Apparatus

Let the set  $\tilde{X}_p$  be partitioned in terms of some specified feature into subsets  $\tilde{X} \subset \tilde{X}_p, |\tilde{X}| = I, I < P, P = |\tilde{X}_p|$ . We call  $\tilde{X}$  the input elements of the  $i$ -th section in the PFR apparatus. Moreover, the elements  $n_\omega$  connected to  $\tilde{X}$  by routes will be referred to as those

